ISSN 2393-8471

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 10, Issue 1, pp: (17-20), Month: April 2023 – September 2023, Available at: www.paperpublications.org

Study of a Fractional Power Series

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong, China

DOI: https://doi.org/10.5281/zenodo.8233144

Published Date: 10-August-2023

Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional power series, we find the exact solution of a fractional power series. In fact, our result is a generalization of traditional calculus result.**

Keywords: **Jumarie type of R-L fractional derivative, new multiplication, fractional power series, exact solution.**

I. INTRODUCTION

Fractional calculus with derivatives and integrals of any real or complex order has its origin in the work of Euler, and even earlier in the work of Leibniz. Shortly after being introduced, the new theory turned out to be very attractive to many famous mathematicians and scientists, for example, Laplace, Riemann, Liouville, Abel, and Fourier. Fractional calculus has important applications in physics, mechanics, biology, electrical engineering, viscoelasticity, control theory, economics, and other fields [1-16].

However, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definition is Riemann-Liouville (R-L) fractional derivatives. Other useful definitions include Caputo fractional derivatives, Grunwald-Letnikov (G-L) fractional derivatives, and Jumarie type of R-L fractional derivatives to avoid non-zero fractional derivative of constant function [17-21].

In this paper, we obtain the exact solution of the following α-fractional power series:

$$\sum_{n=0}^{\infty}\frac{1}{\Gamma(3n\alpha+1)}x^{3n\alpha}\ , \tag{1}$$

where $0<\alpha\leq 1$. Jumarie's modified R-L fractional derivative and a new multiplication of fractional power series play important roles in this paper. Moreover, our result is a generalization of the result in ordinary calculus.

II. PRELIMINARIES

At first, we introduce the fractional derivative used in this paper.

Definition 2.1 ([22])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function. On the other hand, for any positive integer p, we define $\left({}_{x_0}D_x^{\alpha}\right)^p[f(x)]=\left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the p-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([23])**:** *If* α,β,x_0,C *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

ISSN 2393-8471

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 10, Issue 1, pp: (17-20), Month: April 2023 – September 2023, Available at: www.paperpublications.org

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

Definition 2.3 ([24])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is a α-fractional power series at $x = x_0$.

In the following, we introduce a new multiplication of fractional power series.

Definition 2.4 ([25])**:** If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{6}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

Definition 2.5 ([26])**:** If $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{9}$$

On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{10}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{11}$$

III. MAIN RESULT

In this section, we obtain the exact solution of a fractional power series.

Example 3.1: Let $0 < \alpha \leq 1$. Find the α-fractional power series

$$\sum_{n=0}^{\infty} \frac{1}{\Gamma(3n\alpha+1)} x^{3n\alpha}.$$

Solution Let $y_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(3n\alpha+1)} x^{3n\alpha}$, then the α-fractional derivatives of $y_\alpha(x^\alpha)$

$$\left({}_0D_x^{\alpha}\right)[y_\alpha(x^\alpha)] = \sum_{n=1}^{\infty} \frac{1}{\Gamma((3n-1)\alpha+1)} x^{(3n-1)\alpha}, \tag{12}$$

$$\left({}_0D_x^{\alpha}\right)^2[y_\alpha(x^\alpha)] = \sum_{n=1}^{\infty} \frac{1}{\Gamma((3n-2)\alpha+1)} x^{(3n-2)\alpha}, \tag{13}$$

International Journal of Recent Research in Civil and Mechanical Engineering (IJRRCME)
Vol. 10, Issue 1, pp: (17-20), Month: April 2023 – September 2023, Available at: www.paperpublications.org

Thus,

$$\left({}_0D_x^\alpha\right)^2[y_\alpha(x^\alpha)] + \left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)] + y_\alpha(x^\alpha) = E_\alpha(x^\alpha), \tag{14}$$

and

$$y_\alpha(0) = 1, \left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)](0) = 0. \tag{15}$$

This is a second order linear α-fractional differential equation, and we can obtain the general solution is

$$y_{\alpha,h}(x^\alpha) = C_1 E_\alpha\left(-\frac{1}{2}x^\alpha\right) \otimes_\alpha cos_\alpha\left(\frac{\sqrt{3}}{2}x^\alpha\right) + C_2 E_\alpha\left(-\frac{1}{2}x^\alpha\right) \otimes_\alpha sin_\alpha\left(\frac{\sqrt{3}}{2}x^\alpha\right), \tag{16}$$

where C_1, C_2 are constants. On the other hand, the particular solution is

$$y_{\alpha,p}(x^\alpha) = \frac{1}{3}E_\alpha(x^\alpha). \tag{17}$$

Thus,

$$\begin{aligned} & y_\alpha(x^\alpha) \\ &= y_{\alpha,h}(x^\alpha) + y_{\alpha,p}(x^\alpha) \\ &= C_1 E_\alpha\left(-\frac{1}{2}x^\alpha\right) \otimes_\alpha cos_\alpha\left(\frac{\sqrt{3}}{2}x^\alpha\right) + C_2 E_\alpha\left(-\frac{1}{2}x^\alpha\right) \otimes_\alpha sin_\alpha\left(\frac{\sqrt{3}}{2}x^\alpha\right) + \frac{1}{3}E_\alpha(x^\alpha). \end{aligned} \tag{18}$$

By initial value conditions, we have

$$C_1 = \frac{2}{3}, C_2 = 0. \tag{19}$$

That is,

$$\sum_{n=0}^{\infty} \frac{1}{\Gamma(3n\alpha+1)} x^{3n\alpha} = \frac{2}{3}E_\alpha\left(-\frac{1}{2}x^\alpha\right) \otimes_\alpha cos_\alpha\left(\frac{\sqrt{3}}{2}x^\alpha\right) + \frac{1}{3}E_\alpha(x^\alpha). \tag{20}$$

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional derivative and a new multiplication of fractional power series, we find the exact solution of a fractional power series. In addition, our result is a generalization of classical calculus result. In the future, we will continue to use Jumarie type of R-L fractional derivative and the new multiplication of fractional power series to solve problems in applied mathematics and fractional differential equations.

REFERENCES

[1] E. Soczkiewicz, Application of fractional calculus in the theory of viscoelasticity, Molecular and Quantum Acoustics vol.23, pp. 397-404, 2002.

[2] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, Fractals and Fractional Calculus in Continuum Mechanics, pp. 291-348, Springer, Wien, Germany, 1997.

[3] R. Hilfer (ed.), Applications of Fractional Calculus in Physics, WSPC, Singapore, 2000.

[4] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[5] R. L. Magin, Fractional calculus in bioengineering, 13th International Carpathian Control Conference, 2012.

[6] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp. 41-45, 2016.

[7] V. V. Uchaikin, Fractional Derivatives for Physicists and Engineers, Vol. 1, Background and Theory, Vol. 2, Application. Springer, 2013.

[8] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, Vol. 8, No. 5, 660, 2020.

[9] M. F. Silva, J. A. T. Machado, A. M. Lopes, Fractional order control of a hexapod robot, Nonlinear Dynamics, vol. 38, pp. 417-433, 2004.

[10] A. Carpinteri, F. Mainardi, (Eds.), Fractals and fractional calculus in continuum mechanics, Springer, Wien, 1997.

[11] N. Heymans, Dynamic measurements in long-memory materials: fractional calculus evaluation of approach to steady state, Journal of Vibration and Control, vol. 14, no. 9, pp. 1587-1596, 2008.

[12] R. C. Koeller, Applications of fractional calculus to the theory of viscoelasticity, Journal of Applied Mechanics, vol. 51, no. 2, 299, 1984.

[13] T. Sandev, R. Metzler, & Ž. Tomovski, Fractional diffusion equation with a generalized Riemann–Liouville time fractional derivative, Journal of Physics A: Mathematical and Theoretical, vol. 44, no. 25, 255203, 2011.

[14] J. P. Yan, C. P. Li, On chaos synchronization of fractional differential equations, Chaos, Solitons & Fractals, vol. 32, pp. 725-735, 2007.

[15] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[16] C. -H. Yu, A new insight into fractional logistic equation, International Journal of Engineering Research and Reviews, vol. 9, no. 2, pp.13-17, 2021.

[17] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations; John Willy and Sons, Inc.: New York, NY, USA, 1993.

[18] K. B. Oldham, J. Spanier, The Fractional Calculus; Academic Press: New York, NY, USA, 1974.

[19] Podlubny, Fractional Differential Equations; Academic Press: New York, NY, USA, 1999.

[20] S. Das, Functional Fractional Calculus, 2nd Edition, Springer-Verlag, 2011.

[21] Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.

[22] C. -H. Yu, Application of differentiation under fractional integral sign, International Journal of Mathematics and Physical Sciences Research, vol. 10, no. 2, pp. 40-46, 2022.

[23] U. Ghosh, S. Sengupta, S. Sarkar and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, American Journal of Mathematical Analysis, vol. 3, no. 2, pp. 32-38, 2015.

[24] C. -H. Yu, Study of fractional analytic functions and local fractional calculus, International Journal of Scientific Research in Science, Engineering and Technology, vol. 8, no. 5, pp. 39-46, 2021.

[25] C. -H. Yu, Exact solutions of some fractional power series, International Journal of Engineering Research and Reviews, vol. 11, no. 1, pp. 36-40, 2023.

[26] C. -H. Yu, Research on two types of fractional integrals, International Journal of Electrical and Electronics Research, vol. 10, no. 4, pp. 33-37, 2022.